

# An operational view on the holographic information bound

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## Abstract

We study the covariant holographic entropy bound from an operational standpoint. Therefore we consider the physical limit for observations on a *light-sheet*. A light-sheet is a particular null hypersurface, and the natural measuring apparatus is a screen. By considering the physical properties of the screen - as dictated by quantum mechanics - we derive an uncertainty relation. This connects the number of bits of decoded information on the light-sheet to two geometric uncertainties: the uncertainty on the place where the bits are located and the uncertainty on the local expansion of the light-sheet. From this relation we can argue a local operational version of the (generalized) covariant entropy bound: the maximum number of bits decoded on a light-sheet interval goes like the area difference (in Planck units) of the initial and final surface spanned by the light rays.

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# 1 Introduction

The holographic information bound states that, at the fundamental level, the information content of a volume is bounded by the area of its boundary surface at one bit per Planck area [1]. This scaling of information with the surface area rather than the volume, is clearly at odds with any expectation from local quantum field theory. Nevertheless it seems to be a fact of nature. And as such, it appears to reveal an important, yet mysterious aspect of the true nature of (quantum) gravity.

Actually, proclaiming the holographic information bound as a fact of nature, requires a carefull formulation. This formulation was given by Bousso and goes by the name of the covariant entropy bound [2]. It is this bound that has been empirically validated for a variety of physical systems, ranging from black holes and collapsing stars, to entire FLRW universes [3, 4]. To state the covariant entropy bound, one can pick an arbitrary two-dimensional spatial surface. This surface then lies on the boundary of a *light-sheet*  $L$ . This is the null hypersurface, that is generated by surface-orthogonal null geodesics, in the contracting (non-expanding) direction from the boundary surface. A light-sheet (locally) terminates when a caustic or singularity is reached. The covariant entropy bound now states for the entropy  $S$  on  $L$ :<sup>1</sup>

$$S(L) \leq \frac{A}{4l_P^2}. \quad (1.1)$$

So far, no general derivation of this bound has been given. However, Flanagan, Marolf and Wald were able to formulate two sets of general assumptions about the relation between entropy density and energy density, from which the bound can be derived [5]. Furthermore, they showed that one set of their assumptions implied an even stronger bound, for the entropy on general *light-sheet intervals*. Consider a light-sheet that starts off from a surface  $B$  with area  $A$ . Now, instead of following the light rays<sup>2</sup> all the way to the caustics, we can terminate the light-sheet sooner, with the rays spanning a surface  $B'$ , with non-zero area  $A' < A$ . The generalized covariant entropy bound (GCEB) then states for the entropy on this light-sheet interval:

$$S \leq \frac{A - A'}{4l_P^2}. \quad (1.2)$$

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<sup>1</sup>We work in natural units  $c = \hbar = k_b = 1$ , but we will keep track of the Planck length  $l_P = \sqrt{G_N}$  in our equations.

<sup>2</sup>In this paper, we use the term 'light ray' as a synonym for 'null-geodesic', not to denote an actual photon path.

Clearly, the covariant bound (1.1) is a special case of (1.2). And also the stronger bound seems to hold for physical systems. There were some presupposed counterexamples for short light-sheet intervals [3]. But it was shown that these examples signalled a failure of the local description of entropy at short distances, rather than a failure of the GCEB [6].

In this paper we derive an operational version of this stronger entropy bound. For this we need a suitable interpretation of the entropy in (1.2). As was stressed by Bousso, the covariant entropy bound does not single out a preferred time direction [2]. Therefore the entropy in (1.2) has to be truly statistical in nature, referring to the amount of microscopic information [7]. We can therefore interpret  $S = \Delta N$ , as the number of (nat)bits that specify a particular light-sheet interval. From the operational standpoint this becomes the number of bits that can be acquired by physical measurements on the particular light-sheet interval. From some simple arguments that solely depend on elementary quantum mechanics and (classical) gravity we will argue that this number does indeed obey the bound (1.2). As measuring apparatus we will consider a screen. We will later substantiate to some extent that a screen does indeed represent an optimal measuring device for light-sheets.

## 2 The screen

To capture the balance between energy cost and information capacity that is set by quantum mechanics, we will model an idealized screen. This follows largely the construction of [8] (section 3). We consider the screen consisting of  $N_{pix}$  individual pixels with size  $l_{pix}$ . This pixel size, sets both the space and time resolution of the screen. We will sometimes also use  $\Lambda = 1/l_{pix}$  and we can think of this as the energy resolution or the bandwidth of the screen. The screen has a surface area  $A = N_{pix}l_{pix}^2$ , but it will be vital to recognize that it has a thickness as well, of the order  $l_{pix}$ . Furthermore, as a check on the generality of some of our results, we explicitly take into account the number of species  $N_s$  that can encode information. This does not matter much for a screen that is only sensitive to photons for instance, but it will be important when the number of species gets large, like in the case of the 'CFT screen' on AdS [9]. The maximum number of bits that can be decoded on the screen in a time interval  $l_{pix}$  will

be:

$$\Delta N = N_s N_{pix} = N_s \frac{A}{l_{pix}^2}. \quad (2.3)$$

As most of our equations from now on, this has to be interpreted as an estimate, for which the precise (order one) pre-factor is undetermined.

For each pixel, the screen needs the ability to recognize each of the  $N_s$  species. This can be achieved for instance, by storing  $N_s$  sample particles in every pixel, that can interact with - and therefore detect - every type of particle. The minimal energy of a sample particle localized within a pixel of size  $l_{pix}$ , is  $E = 1/l_{pix}$ , so every pixel will necessarily store an energy  $E = N_s/l_{pix} = N_s \Lambda$ . Conversely, the energy of a sample particle sets a minimal thickness of the pixel - and therefore the screen - of the order  $l_{pix}$ , as we mentioned above. Furthermore, neglecting boundary effects, we can write down an expression for the (coarse-grained) energy-momentum tensor of the pixel:

$$T_{\mu\nu} = N_s \Lambda^4 (U_\mu U_\nu - k g_{\mu\nu}), \quad (2.4)$$

where  $U_\mu$  is the four-velocity of the pixel (screen) and with  $k$  an order one constant.

Finally, as was argued in [8], the required energy for the pixel sets an intrinsic maximum resolution. Indeed, when the energy of the pixel becomes too large, it collapses to a black hole. This results in the bound:

$$l_{pix}^2 \geq l_{N_s}^2 \equiv N_s l_P^2. \quad (2.5)$$

For a few species, this is simply the operational derivation of  $l_P$  as the smallest length scale; for a large number of species this is the hierarchy between the smallest length scale  $l_{N_s}$  at which individual species can be resolved and the Planck length.

### 3 Observing on the light-sheet

We set things up, so that we can use our screen to make a measurement on a particular light-sheet. This can be done in an obvious way, by positioning and orienting the screen such that the light rays of the light-sheet at some value  $\lambda$  of the affine parameter, will be orthogonal to the screen at some time  $t$  in the screen reference frame. At this instant, the screen coincides with part of the surface  $B(\lambda)$  spanned by the light rays of the sheet. Notice that we are assuming that the size of our screen is smaller than the smallest distance scale set by the intrinsic and extrinsic curvature of the surface on the

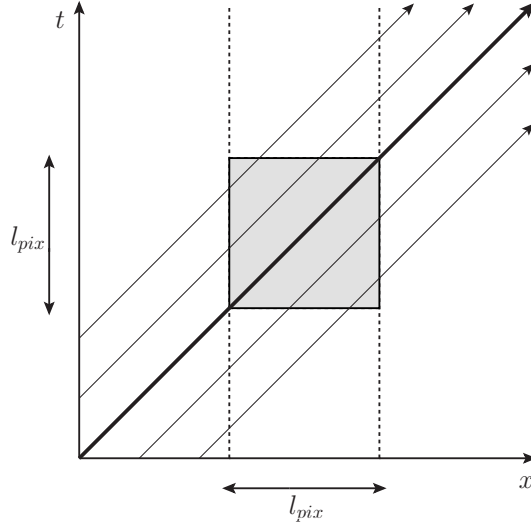


Figure 1: A measurement in the screen rest frame, with the  $x$ -direction orthogonal to the screen. The measurement takes place in the shaded space-time region set by the pixel size. Correspondingly, we will consider the acquired information to be located on the coarse-grained light-sheet (depicted as a thick light ray), over an interval  $\Delta\lambda$ .

light-sheet, so that we can work in the local flat limit. Since a screen may consist of only a few pixels, this is in fact a condition on the pixel size  $l_{pix}$ . Now, if the measurement would be really instantaneous and the screen would be truly a two-dimensional object, we could assign all the information acquired during the measurement to (part of) the surface  $B(\lambda)$  of the specific light-sheet. But clearly this is not the case. As we discussed above, the screen necessarily has a non-zero space-time resolution  $l_{pix}$ , and a thickness of the same order. As illustrated in the figure, this carries through to a smallest interval  $\Delta\lambda$  of the affine parameter, over which we can locate the measured information on the *coarse-grained* light-sheet. We will not further specify the coarse-graining procedure, and leave it at the simple picture of the figure. In fact, in what follows we will drop the adjective and simply talk of the light-sheet. With  $k^\mu = dx^\mu/d\lambda$ , the local tangent vector to the light rays (and  $U^\mu$  the four-velocity of the screen), the covariant expression for the light-sheet interval reads:

$$\Delta\lambda = \frac{l_{pix}}{|k_\mu U^\mu|}. \quad (3.6)$$

This holds at first order in  $\Delta\lambda$ , as will be the case for the other derivations that follow.

Since we are interested in the maximum number of bits that can be decoded, we will estimate the maximum possible resolution of our screen. Of course, we first of all

have the intrinsic bound (2.5), set by the screen itself. But, as we will now argue, there is another bound, that is set by the particular light-sheet that we want to observe. It is the latter bound that will translate to an operational version of the GCEB.

Our heuristic argument on this bound follows the mantra of the classic quantum mechanics gedanken experiments on the uncertainty relations. Namely, that one should always consider the influence of the measuring apparatus (in our case the screen) on the object being measured (in our case the light-sheet). Clearly, the screen has energy, energy bends light rays, therefore the light-sheet is disturbed by the screen. To quantify this effect we should look at the expansion of the family of light rays that generate the light-sheet. The expansion  $\theta$  of a family of light rays with tangent vector  $k^\mu$  is defined as (see e.g. [3, 10]):

$$\theta(\lambda) = \nabla_\mu k^\mu = \frac{1}{\mathcal{A}} \frac{d\mathcal{A}}{d\lambda}, \quad (3.7)$$

where  $\mathcal{A}$  is the surface area spanned by the infinitesimally neighboring light rays. Its evolution is governed by the (zero twist) Raychaudhuri equation:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 8\pi G_N T_{\mu\nu} k^\mu k^\nu. \quad (3.8)$$

The second term on the right-hand side is always non-positive, and gives the focussing contribution of the shear. The third term is non-positive if matter obeys the *null energy condition*,  $T_{\mu\nu} k^\mu k^\nu \geq 0$ . This term then captures the focussing effect of the total energy-momentum and the screen influences the expansion in the first place through its contribution to this term.<sup>3</sup> The screen contribution is nonzero only in the interval  $\Delta\lambda$ , and we can estimate the resulting 'kick'  $\Delta\theta$  of the expansion by plugging in the expression (2.4) for the energy-momentum tensor of the screen and using our expression (3.6) for  $\Delta\lambda$ :

$$\Delta\theta = -N_s l_P^2 \Lambda^4 (k_\mu U^\mu)^2 \Delta\lambda = -\frac{N_s l_P^2 |k_\mu U^\mu|}{l_{pix}^3}. \quad (3.9)$$

We can now interpret  $-\Delta\theta$  as the uncertainty on the expansion of the light-sheet, associated with the screen measurement. Inspection of the full equation (3.8) shows that, at best, this estimate holds for  $\Delta\lambda \leq 1/|\theta|$ , with  $\theta$  the undisturbed expansion. Notice that this range also corresponds to the maximal possible range on the light-sheet, since by the *focussing theorem* we have that  $\theta(\lambda + \Delta\lambda) \rightarrow -\infty$  for  $\Delta\lambda \leq 2/|\theta(\lambda)|$ , as one can easily show from (3.8) (see e.g. [3]).

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<sup>3</sup>The screen also influences the evolution of the shear through its contribution to the Weyl curvature. But one can show that the resulting local effect on  $\Delta\theta$  is only of order  $\Delta\lambda^2$ .

So we find three relevant quantities that depend on the screen resolution, velocity and number of decoding species. There are the uncertainties on  $\theta$  (3.9) and on the value of the affine parameter  $\lambda$  (3.6). And there is the maximum number of bits  $\Delta N$  (2.3) that can be acquired in the measurement. Taken together, we can write this interdependence in the following uncertainty relation (dropping the minus sign in (3.9)):

$$\Delta\theta \cdot \Delta\lambda \geq \frac{l_P^2}{A} \Delta N, \quad (3.10)$$

the uncertainty on the local expansion multiplied by the uncertainty on the affine parameter of the light-sheet is larger than the number of bits decoded on the corresponding light-sheet interval, per spanned surface area in Planck units. Note that this relation is covariant and does not depend on the particular affine parameterization. Nor does it depend on the different screen properties, which suggests a universal character.

It is now a small, albeit heuristic step from the uncertainty relation above, to an operational version of the GCEB (1.2). In the same way that one argues from Heisenberg's uncertainty principle, that  $1/m$  is the smallest wavelength that can be used to resolve a particle with mass  $m$ ; we can now argue the largest resolution allowed for the screen to observe at a particular place of the light-sheet. We simply require the measurement kick or uncertainty on the local expansion, to be less than the expansion itself. This leads to the condition:

$$\Delta\theta = |\theta|, \quad (3.11)$$

for the maximum resolution of the screen observing on a light-sheet with expansion  $\theta$  at the cross-section with the screen. For observations at this maximum resolution, the uncertainty relation (3.10) becomes:

$$\Delta N \leq \frac{A}{l_P^2} |\theta| \Delta\lambda = \frac{1}{l_P^2} \Delta A, \quad (3.12)$$

the maximum number of bits that can be decoded on a light-sheet interval is smaller than the change in the surface area spanned by the light rays, in Planck units. Apart from the constant  $1/4$ , that can not be determined from our argument, we consider this to be the *local* operational version of the GCEB (1.2). Local in the sense that our derivation holds up to first order in  $\Delta\lambda$ . As such, our operational construction is also too rudimentary to go beyond this order.

For the validity of the covariant entropy bound one usually assumes two energy conditions on the matter sector [3]. First, there is the null energy condition that we quoted before. Secondly, one assumes the *causal energy condition* which excludes superluminal propagation of energy. In our derivation of the operational bound we have implicitly used the very same two energy conditions. Indeed, when modelling the screen, we assumed that there were no *ghosts*, i.e. quanta with negative energy, that are excluded by the null energy condition. Those particles would make it possible to store information without the usual energy cost. In addition, when converting the spatial resolution to a time resolution we use the speed of light. This would obviously change in the case of superluminal propagation.

Furthermore, the covariant entropy bound is considered to be a classical bound in the sense that it only applies to those regimes where the space-time geometry can be treated as approximately classical. This is also the case for our operational bound. But notice that the bound itself follows precisely from recognizing that the concept of absolute classical space-time geometry does not make sense from an operational point of view. Indeed, according to our uncertainty relation (3.10), one can not both determine the local expansion and the corresponding affine parameter up to an arbitrary accuracy.

The physical mechanism behind the operational bound is clear. A measurement that allocates  $\Delta N$  bits of information to a certain place on the light-sheet, necessarily comes with an uncertainty  $\Delta\lambda$  on the precise location of these bits, such that the corresponding light-sheet interval obeys the bound (3.12). To see in detail how this comes about for measurements with our screen, we have to look at the maximum screen resolution, at which the bound (3.12) can be saturated. We will call this *the holographic resolution*. From the condition (3.11) and the expression (3.9) for  $\Delta\theta$ , we find

$$l_{pix}^3 = l_{hol}^3 \equiv \frac{N_s l_P^2 |k_\mu U^\mu|}{|\theta|}, \quad (3.13)$$

at this maximum resolution, for a screen with  $N_s$  species and four-velocity  $U^\mu$ , observing on a light-sheet locally generated by  $k^\mu$ , with expansion  $\theta$ . As before, this formula is covariant and independent of the particular affine parameterization. But notice that it does depend on the screen velocity. By varying the speed of the screen in the direction of the light-rays from  $c$  to  $-c$ , the factor  $|k_\mu U^\mu|$  varies from zero to infinity. Suppose now we have a screen with resolution  $l_{pix}$  and we want this screen to operate at the holographic resolution for a given light-sheet with local expansion  $\theta$ . This requires a



screen velocity such that:

$$|k_\mu U^\mu| = \frac{|\theta|}{N_s l_P^2} l_{pix}^3, \quad (3.14)$$

and for the resolution  $\Delta\lambda$  of the light-sheet position we then find:

$$\Delta\lambda = \frac{l_{pix}}{|k_\mu U^\mu|} = \frac{N_s l_P^2}{|\theta| l_{pix}^2}. \quad (3.15)$$

The maximum number of decoded bits  $\Delta N$  (2.3) then indeed obeys the operational bound (3.12). It is now interesting to look what happens for a screen that operates at maximum intrinsic resolution  $l_{pix}^2 = N_s l_P^2$ . As one sees from (2.3), such a screen can acquire one bit per Planck area in a measurement. This corresponds to Bousso's bound for entire light-sheets. Our estimate (3.15) of the associated light-sheet interval then reads  $\Delta\lambda = 1/|\theta|$ . Which, as we discussed earlier, corresponds to the maximum possible range for which our derivation holds and which is also simply the maximum possible range on the light-sheet.

Clearly, the most heuristic step in our derivation of the operational information bound is the condition (3.11), that determines the holographic resolution. Let us briefly illustrate its physical meaning for the simplest example. We consider (static) spherical screens that observe on the light-sheets emanating from some spherical surface in Minkowski space-time. Substituting the expansion and velocity factor in (3.13), we find the holographic resolution  $l_{pix}^3 = N_s l_P^2 r$  for a screen at a distance  $r$  from the origin. The screen has a thickness  $l_{pix}$  and therefore a volume  $r^2 l_{pix}$ . Using the expression (2.4) for the density,  $\rho = N_s / l_{pix}^4$ , this leads to a mass  $M = r / l_P^2$ . So in this case, our condition (3.11) simply states that the screen itself should not be a black hole. In [9] we will provide further illustrations on other space-times.

Of course, our derivation of the general uncertainty relation (3.10) and the corresponding operational bound (3.12), also crucially hinges on the assumption that the screen is an optimal measuring apparatus for light-sheets. We can not prove this. However, we can consider a slight variation of our screen and show that it performs worse, in the sense that it can not saturate the bound (3.12). Let us take a multi-layered screen consisting of a stack of  $n$  ordinary screens. Both the kick in the expansion  $\Delta\theta$ , as the interval of the affine parameter  $\Delta\lambda$  (but not the resolution), as the number of bits  $\Delta N$  per measurement, scales linearly with  $n$ . Following the same logic that we applied before, this leads to an extra factor  $1/n$  on the right-hand side of the inequality (3.12), for observations with this multi-layered screen.

Finally, the generalization of our results to arbitrary  $d + 1$  space-time dimension is straightforward. The uncertainty relation (3.10) and the bound (3.12) for example, remain of the same form, with on the right-hand side,  $A/l_P^{d-1}$  denoting the surface area of the relevant codimension 2 surface in Planck units. While the formula (3.13) for the holographic resolution becomes:

$$l_{hol}^d = \frac{N_s l_P^{d-1} |k_\mu U^\mu|}{|\theta|}. \quad (3.16)$$

## 4 To Conclude

In this paper we have in some sense provided an approximate *derivation* of the (generalized) covariant entropy bound. But even if we could go beyond our local approximation, and even if we could determine the precise pre-factors, we would not consider this to constitute an *explanation*. The holographic entropy bound is such a beautiful relation that it would be surprising if it's merely a consequence of quantum mechanics and classical gravity. One would expect there to be new physics. The case in point being the AdS/CFT connection. Maybe our operational approach will help in unveiling some of the new physics for other, more general space-times. In this light we can not resist to point out the analogy with the relation between Bekenstein's entropy bound [11] and Heisenberg's uncertainty relation. In that case one can interpret the bound as the information theoretic translation of the uncertainty relation [12]. In the same way one could view the generalized covariant entropy bound as the translation of our uncertainty relation (3.10).

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## References.

- [1] G. 't Hooft, arXiv:gr-qc/9310026; L. Susskind, J. Math. Phys. **36** (1995) 6377 [arXiv:hep-th/9409089].

- [2] R. Bousso, JHEP **9907** (1999) 004 [arXiv:hep-th/9905177].
- [3] R. Bousso, Rev. Mod. Phys. **74** (2002) 825 [arXiv:hep-th/0203101].
- [4] R. Bousso, B. Freivogel and S. Leichenauer, arXiv:1003.3012 [hep-th].
- [5] E. E. Flanagan, D. Marolf and R. M. Wald, Phys. Rev. D **62** (2000) 084035 [arXiv:hep-th/9908070].
- [6] R. Bousso, E. E. Flanagan and D. Marolf, Phys. Rev. D **68**, 064001 (2003) [arXiv:hep-th/0305149].
- [7] E. T. Jaynes, Phys. Rev. **106** (1957) 620.
- [8] G. Dvali and C. Gomez, Phys. Lett. B **674** (2009) 303 [arXiv:0812.1940 [hep-th]].
- [9] K. Van Acoleyen, Work in progress.
- [10] R.M. Wald, "General Relativity," The University of Chicago Press, Chicago
- [11] J. D. Bekenstein, Phys. Rev. D **23** (1981) 287.
- [12] R. Bousso, JHEP **0405** (2004) 050 [arXiv:hep-th/0402058].